Critical behavior of the two-dimensional spin-diluted Ising model via the equilibrium ensemble approach

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The equilibrium ensemble approach to disordered systems is used to investigate the critical behavior of the two-dimensional Ising model in the presence of quenched random site dilution. The numerical transfer matrix technique in semi-infinite strips of finite width, together with phenomenological renormalization and conformal invariance, is particularly suited to putting the equilibrium ensemble approach to work. A method by which to extract with great precision the critical temperature of the model is proposed and applied. A more systematic finite-size scaling analysis than in previous numerical studies has been performed. A parallel investigation, along the lines of the two main scenarios currently under discussion, namely, the logarithmic corrections scenario (with critical exponents fixed in the Ising universality class) versus the weak universality scenario (critical exponents varying with the degree of disorder), is carried out. In interpreting our data, maximum care is constantly taken to be open to both possibilities. A critical discussion shows that an unambiguous discrimination between the two scenarios is still not possible on the basis of the available finite-size data.
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I. INTRODUCTION

Recent years have witnessed renewed efforts within the statistical physics community to understand phase transitions in simple disordered classical spin systems. As a matter of fact, these efforts have produced conflicting statements concerning the effects of disorder on critical phenomena, covering almost the complete spectrum of conceivable alternatives.

This holds in particular, but not exclusively, for two-dimensional (2D) disordered ferromagnetic (i.e., unfrustrated) Ising models. These models have been widely studied, both because the corresponding pure system is well understood and because they constitute a marginal case in the Harris criterion [1], which assesses whether disorder constitutes a relevant or irrelevant perturbation for the critical behavior of the pure system. For models of this type, the discussion currently appears to narrow down to two conflicting scenarios, namely, the *logarithmic corrections* [2–17] versus the *weak universality* [18–21] scenario, although a broader spectrum of alternatives had been discussed earlier [22–24] (the interested reader will find a comprehensive report on the literature up to approximately 1982 in an early review by Stinchcombe [25]). We will describe these scenarios and discuss these (and related) results in greater detail later on.

The object of our study is the randomly spin diluted 2D Ising model. Our investigation is based on two main ingredients. First, we use the equilibrium ensemble approach to disordered systems [26,27,18,21,28] to map the quenched system onto an equivalent thermodynamic equilibrium system in an enlarged phase space. Performing this mapping exactly would be tantamount to providing an exact solution to the original problem, which is clearly unfeasible. So a scheme of approximations based on a moment matching idea is invoked. Second, we resort to conventional transfer-matrix (TM) techniques to implement the method in practice on finite-width strips, analyzing finite-size results along the lines of Nightingale's phenomenological renormalization group scheme [29,30].

The purpose of the present paper is to provide details of our TM study [18,21] as well as to include new material and to describe significant advances in the understanding of finite-size scaling (FSS) signatures in the presence of logarithmic corrections, which together have allowed us to boost the accuracy of our results considerably and to obtain a sounder appreciation of the subtleties that may emerge in the interpretation of the data.

The main and unexpected finding in Ref. [21] has been a continuous variation of the critical exponents α, β, γ, and ν with the spin density ρ in a manner which was observed to comply with the idea of weak universality [31]. That is, the exponent η describing the decay of critical correlations, and the magnetic exponent δ, as well as the ratios β/ν and γ/ν were found to be independent of ρ. The results were obtained by extrapolations of FSS data based on rather moderate strip widths, and they were in complete quantitative and qualitative agreement with those of a Monte Carlo study by Kim and Patrascioiu [20]. These results are in conflict with those supporting the logarithmic corrections scenario, where modifications of the relevant thermodynamic quantities at criticality appear through logarithmic terms in the reduced temperature, while the system is left in the same universality class (i.e., with the same critical exponents) as that of the pure 2D Ising model. These findings almost all concern the ferromagnetic-bond-disordered situation, either theoretically [3–5,17] or via numerical [6,7,9–14,16] and series expansion [15] approaches, but the comparison with the spin-diluted model is possible assuming—as is generally believed

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to be the case—that the two systems are in the same universality class [25,17].

In discussing our results, we have to address two main issues. The first is concerned with the reliability of our method, which is based on an approximate description of quenched disorder. The evidence we have been able to compile does give us strong confidence in the validity of our approach. This granted, we turn to the second issue: can—or, more precisely, to what extent can—our results provide evidence in favor of or against any of the conflicting scenarios so far advanced to describe the critical behavior of 2D disordered ferromagnetic Ising models? This is indeed a subtle question, and we devote almost two Sections (Secs. IV and V) to discussing it. It turns out that a considerable portion of the available finite-size data from simulations or TM studies—including those presented in our earlier study [18,21] as well as some new ones—may perhaps *not* allow one to decide with sufficient confidence between the two most serious candidates. And we shall explain why. Larger system sizes would in any case be necessary in studies along those lines to permit taking sides.

Finally, since one of the aims of our study is to discriminate between these two contradicting scenarios, one of our main concerns is an open-minded attitude, perhaps not easily found in the literature, in the analysis of the numerical results: we take the greatest care not to select one of them *a priori* and to fit our data to it, but, rather, we attempt to be open to both possibilities.

The remainder of our paper is organized as follows. In Secs. II A and II B, in order to set the scene and to fix our notation, we describe the model that is studied in the sequel and we briefly recall our methodological background viz. the equilibrium ensemble approach (EEA) to disordered systems. While Sec. II C presents some details related to the computational side of the problem, we reserve Sec. III for a brief review of the two above-mentioned scenarios, and Sec. IV for a presentation and discussion of our main results. We describe a finite-size procedure to extract the critical temperature of the model as a function of the degree of dilution, which enables us to draw an accurate approximation of the phase diagram. We then determine values for the critical exponents, for the central charge, and for the specific heat of our approximating systems. We end our paper with a critical report of the available finite-size data obtained also by other authors, and with a comprehensive discussion both on the method and on the outcome of our investigation (Sec. V). An Appendix, mainly based on renormalization-group results by Cardy and Ludwig [32,33] presents a systematic FSS analysis of thermodynamic quantities in the presence of logarithmic corrections.

II. SETTING THE SCENE

A. The system under investigation

The object of our investigation is the 2D randomly spin-diluted Ising model, described by the Hamiltonian

$$\mathcal{H}(\sigma|\kappa) = -J\sum_{\langle ij\rangle} k_i\sigma_i k_j\sigma_j - H\sum_i k_i\sigma_i . \tag{1}$$

Here, the first sum is over all nearest-neighbor pairs $\langle ij\rangle$ of, say, a square lattice Λ, and the second over all sites i. The σ_i denote Ising spins and the $k_i \in \{0,1\}$ are occupation numbers signifying whether in a disorder configuration κ a site i is occupied by a spin ($k_i=1$) or is not ($k_i=0$). The k_i are taken to be quenched random variables, i.e., they are fixed, and randomly chosen according to the probability

$$q(\kappa) = \prod_i \rho^{k_i}(1-\rho)^{1-k_i}, \tag{2}$$

which simply requires each site to be occupied with probability ρ, and to be empty with probability $1-\rho$. Thus ρ is the average density of spins in the system.

Our aim is to study the thermodynamics of the system, described by the (dimensionless) quenched free energy

$$f_q = -N^{-1}\langle \ln Z_N(\kappa)\rangle_q , \tag{3}$$

i.e., the average of the system's free energy over the distribution $q(\kappa)$ describing the statistics of the disorder configurations κ. Here N denotes the system size and $Z_N(\kappa)$ is the partition function of a system of size $|\Lambda| = N$ at fixed disorder configuration κ. In particular, we would like to locate the phase-transition line $T_c(\rho)$ below which the system is known to develop spontaneous ferromagnetic order in the thermodynamic limit $N\to\infty$, and to study the critical behavior of the system at and around this line. Whereas the thermodynamics of the 1D system is known exactly, an exact solution of the 2D model is currently way beyond our abilities. Approximation methods, which may, however, well produce exact results in certain limiting cases, are thus called for.

B. Methodological background: The equilibrium ensemble approach

The EEA to disordered systems, as well as its application to the 2D spin-diluted Ising model, have been accurately described elsewhere [28,21]. Therefore, we will not enter again into the details of the method, but just concisely recall the main ideas.

The central idea is to treat the configurational degrees of freedom k_i in a system with quenched disorder on the same footing as the dynamical variables proper—the spins, in the case at hand. That is, the phase space is enlarged to include the occupation-number configurations κ, and an additional term depending only on κ, the so-called *disorder potential* $\phi(\kappa)$, is added to the Hamiltonian

$$\mathcal{H}(\sigma|\kappa) \to \mathcal{H}^{\phi}(\sigma,\kappa) = \mathcal{H}(\sigma|\kappa) + \phi(\kappa), \tag{4}$$

which is then determined in such a way that configuration averaging, as implied by Eq. (3), becomes part of the Gibbs average in the enlarged phase space.

To utilize the method in practice, one still needs an explicit representation for $\phi(\kappa)$. For spin or site diluted systems the following representation in terms of products of occupation numbers

$$-\beta\phi(\kappa)=\lambda_1\sum_i (k_i-\rho)+\lambda_2\sum_{\langle ij\rangle}(k_ik_j-\rho^2)+\cdots$$
$$+\lambda_P\sum_P\left(\prod_{i\in P}k_i-\rho^{|P|}\right)+\cdots \tag{5}$$

is always suitable. The first sum explicitly displayed in Eq. (5) is over all lattice sites, the second over all nearest-neighbor pairs, the third over all elementary plaquettes (of size $|P|$) of the system, and so on.

It has been realized [28] that the self-consistent constraint equations that determine the couplings λ_ν as a function of temperature T, magnetic field H, and spin density ρ (Eqs. (4) of Ref. [21]), can be written as necessary conditions describing the *maximum* of the equilibrium ensemble's dimensionless free energy (per site)

$$f^\phi=-N^{-1}\ln Z^\phi=-N^{-1}\ln\sum_{\sigma,\kappa}\exp[-\beta\mathcal{H}^\phi(\sigma,\kappa)]$$

with respect to the λ_ν's. They can therefore be written, and are in practice evaluated too, as

$$-\frac{\partial f^\phi}{\partial\lambda_1}=\langle k_i\rangle_\phi-\rho=0,$$
$$-\frac{\partial f^\phi}{\partial\lambda_2}=\langle k_ik_j\rangle_\phi-\rho^2=0,... \tag{6}$$
$$-\frac{\partial f^\phi}{\partial\lambda_P}=\left\langle\prod_{i\in P}k_i\right\rangle_\phi-\rho^{|P|}=0,... .$$

Solving Eqs. (6) for every cluster of connected sites—i.e., exactly obtaining the quenched free energy via the EEA—is clearly unfeasible. However, a systematic scheme of approximations may be put up by matching only a *subset* of the full set of moments of the equilibrium ensemble's Gibbs distribution to the corresponding subset of moments of the problem with quenched disorder. This amounts to forcing k_i correlations on larger and larger groups of lattice sites to coincide with those of the quenched system. The larger the number of constraints taken into account, the better a description of the fully quenched system is obtained.

In Ref. [21] four different approximating systems named $(a)-(d)$ were studied, differing by the number of terms kept in Eq. (5), and thus by the set of constraints which is actually retained in Eq. (6). The present study is focused only on approximation (d): the first three contributions to the disorder potential are kept, i.e., single site, pair, and square plaquette terms. In addition, due to the anisotropy of the strip geometry employed in the calculation (see Sec. II C below), pair occupancy parallel and perpendicular to the strip are treated as separate constraints, i.e., they are controlled by separate couplings $\lambda_{2\parallel}$ and $\lambda_{2\perp}$ in the disorder potential which, in turn, have to be determined by two distinct equations of the type (6).

C. Transfer-matrix approach

The approach outlined above might—at least in principle—solve our problem to any desired degree of accuracy. However, in $d\geqslant 2$ not even the simplest approximating system is exactly solvable. To analyze the thermodynamics or the critical behavior of our systems, we use the strip FSS techniques pioneered by Nightingale [29]. A distinct advantage of the EEA in this context is that we are dealing with *translationally invariant equilibrium systems* with short-range couplings at all levels of approximation within the moment-matching scheme described above. FSS can therefore be implemented using conventional TM techniques for nonrandom systems.

Our strip is a $L\times L'$ square lattice, with the thermodynamic limit taken in the L' direction, and periodic boundary conditions (unless otherwise stated) imposed in the perpendicular direction. The row-to-row transfer matrix Γ, constructed from the local Boltzmann factors, can be chosen to be symmetric for all systems $(a)-(d)$ of Ref. [21], and is diagonalized for strips of fixed width L, with the appropriate set of constraints [Eqs. (6)].

The dimensionless free energy f_L (per site) is related to the largest eigenvalue γ_1 of Γ via

$$-f_L=L^{-1}\ln\gamma_1 \tag{7}$$

(to simplify notation, we omit stating the L dependence of Γ and its eigenvalues in what follows).

The constraint equations (6) have been formulated in terms of first derivatives of the free energy, and thereby in terms of first derivatives of f_L. As mentioned above, those derivatives are nothing but gradient information in the problem of maximizing f_L (or minimizing $-f_L$) over the appropriate set of couplings. Note that the transcendental nature of these equations necessitates an iterative numerical solution. Having in this way determined the (approximate) disorder potential, one may compute thermodynamic functions and, in particular, the correlation length of the order-parameter fluctuations, which is needed in the phenomenological renormalization-group scheme. The correlation length is given in terms of the largest and the second largest eigenvalues of the TM as

$$\xi_L^{-1}=-\ln\frac{\gamma_2}{\gamma_1}. \tag{8}$$

Thermodynamic functions of interest are obtained by differentiating the free energy with respect to temperature or magnetic field. Here, care must be taken because of implicit field and temperature dependences in the couplings of the disorder potential. In other words, derivatives have to be performed *along the solution-manifold of the appropriate set of constraint equations*.

Introducing modified spins $S_i=k_i\sigma_i$ with values in $\{0,\pm1\}$, we encode both the presence or absence of a spin (the occupation numbers being given by $k_i=|S_i|$) and the spin states in a single variable. Since the dimension of Γ grows as $3^L\times3^L$, we have used two different strategies—with complementary strengths and weaknesses—to simplify the computational task of dealing with huge transfer matrices. The first is a group-theoretical analysis that exploits the various symmetries of the matrix and effectively reduces its dimensionality. The second is the use of sparse matrix techniques.

We will not bother the reader by giving a detailed account of these techniques since, apart from the fact that they have been nontrivially specialized to deal with our particular system, they are not new (for the sparse matrix technique, see, e.g., Ref. [30]). We just note that within either of them it is possible to identify blocks $\Gamma^{(1)}$ and $\Gamma^{(2)}$, in the block diagonal representation of Γ, that contain its two largest eigenvalues γ_1 and γ_2, respectively. $\Gamma^{(1)}$ is the block spanned by the space of eigenvectors transforming symmetrically under spin reversal, while γ_2 is found in the block spanned by the eigenvectors of Γ which are antisymmetric under spin reversal. In a similar spirit, one may investigate the long distance behavior of the k_i correlation functions $G_k(r_{ij})=\langle k_i k_j\rangle - \langle k_i\rangle\langle k_j\rangle$. This requires locating a corresponding eigenvalue $\tilde{\gamma}_2$, which controls the asymptotic decay of $G_k(r)$. It is found to be the second largest eigenvalue in the block corresponding to a representation of the symmetry group which is *symmetric* with respect to both external (spatial) and internal (spin) symmetries of the system. Indeed, whereas the spins σ_i change sign under global spin reversal, entailing that the eigenvalue to be inserted in Eq. (8) when computing the spin spin correlation length must belong to the antisymmetric block of Γ, the disorder variables $k_i=|S_i|$ do not change sign.

The computation of second-order derivatives of γ_1, needed to evaluate some of the relevant thermodynamical quantities, is feasible only in the group-theoretical reduction approach. The second derivatives in fact require knowledge of the complete eigensystem, which is still far too large within the sparse matrix approach. In this case one has to resort to finite differences of first-order derivatives to compute, e.g., the specific heat or the spin susceptibility. The advantage of the sparse-matrix approach, on the other hand, is that it can be pushed to considerably larger strip widths than the group-theory approach. We reach the value $L=9$ with the latter and $L=13$ with the former. Moreover, this approach does not exploit any symmetries beyond the spin reversal symmetry in zero magnetic field. It is therefore also applicable if one introduces an antiferromagnetic seam along the strip, a device which can be used to compute the domain-wall free energy. We shall exploit this quantity below to considerably boost the precision of our FSS analysis.

III. THE TWO SCENARIOS

Before showing our results, it is time to briefly present the two main scenarios that have survived out of a wider range of possibilities for the description of the critical behavior of the 2D site diluted Ising model, but which are obviously mutually exclusive.

The *logarithmic-corrections* scenario is based on the quantum-field-theory results by Dotsenko and Dotsenko [2] and by Shalaev [3], Shankar [4], and Ludwig [5] (the latter contributions correcting certain errors in the former). Though strictly valid only in the limit of weak disorder, they indicate that the presence of impurities affects the critical properties of the model only through a set of logarithmic corrections to the pure system behavior. In particular, according to this picture, the correlation length of the infinite system close to the phase transition is expected to show the form

$$\xi_\infty \sim t^{-\nu}\left[1+\tilde{g}\ln\left(\frac{1}{t}\right)\right]^{\tilde{\nu}}, \tag{9}$$

where t is the reduced temperature $t=(T-T_c)/T_c \ll 1$ (T_c being the critical temperature, which will depend on ρ), the exponents ν and $\tilde{\nu}$ are, respectively, 1 and $\frac{1}{2}$, and $\tilde{g}$ is a nonnegative constant such that $\tilde{g}=0$ in the pure case (usual power-law behavior recovered) and increases with increasing disorder. A similar behavior holds for the magnetic susceptibility:

$$\chi_\infty \sim t^{-\gamma}\left[1+\tilde{g}\ln\left(\frac{1}{t}\right)\right]^{\tilde{\gamma}}, \tag{10}$$

with $\gamma=\frac{7}{4}$ and $\tilde{\gamma}=\frac{7}{8}$. For the specific heat, too, the exponent of the pure case $\alpha=0$ is not modified by the introduction of disorder, but the simple logarithmic behavior $C_\infty \sim \ln(1/t)$ is replaced by the double logarithmic singularity

$$C_\infty \sim \ln\left[1+\tilde{g}\ln\left(\frac{1}{t}\right)\right]. \tag{11}$$

The behavior of the critical spin-spin correlation function is instead predicted not to change in the presence of impurities, thus its anomalous dimension η retains its pure Ising value $\eta=\frac{1}{4}$.

Recent numerical works [19–21] have provided evidence contradictory to what has just been presented above. These findings show quantities such as the susceptibility and the correlation length to display simple power-law singularities, at critically, with exponents γ and ν varying continuously with disorder in a way, however, that their ratio γ/ν is kept constant at the pure system value. According to this *weak-universality* [31] scenario, the ratio of exponents should not depend on disorder, and the specific heat was observed to saturate at a nondivergent value as $t\to 0$.

In a finite-size numerical investigation like ours, it is also essential to translate the predictions of these scenarios into their finite-size counterparts. This is straightforward and well known in the case of power-law singularities (and power-law corrections to scaling), but less trivial within the logarithmic-corrections scenario. These questions are dealt with in detail in the Appendix and, where they arise, in the following section.

As already stressed in the Introduction, we have tried to look at our results with an open mind: none of the two scenarios is chosen as a reference; rather, we attempt to let our analysis determine which of the two consistently fits the whole series of our data.

IV. MAIN RESULTS

A. Determination of the critical temperature

The 2D spin-diluted Ising model exhibits a phase transition from a paramagnetic to a ferromagnetic low-temperature phase, provided the spin density ρ is larger than the critical site percolation density $\rho_c \simeq 0.592745$ [34]. The model unfortunately lacks the ability, known for its random-bond counterpart, to exactly determine the critical temperature as a function of the degree of disorder from duality. This is one of the factors that render the bond-disordered version more

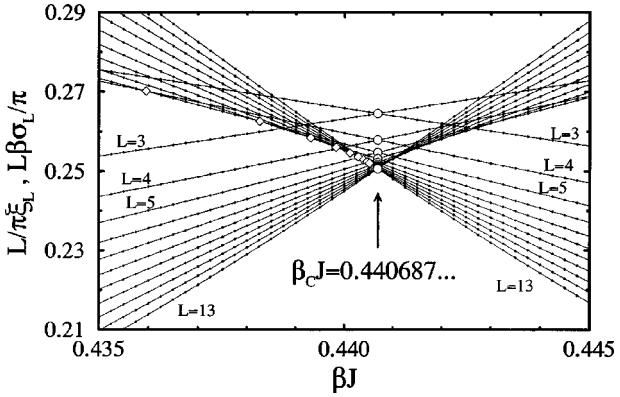

FIG. 1. Set of curves $L/\pi\xi_L$ (negative slope) and $L\beta\sigma_L/\pi$ (positive slope) as a function of βJ for different values of L, for the pure Ising model. Their mutual intersections, at the abscissas $(\beta J)_L^{\rm int}$, are indicated by open circles; the crossings of two $L/\pi\xi_L$ curves for pairs of values L, $L+1[(\beta J)_L^{\xi}]$ are drawn as open diamonds; black dots denote the points used in the temperature scan. It is evident how the former sequence is much more rapidly converging than the latter to the exact value $\beta_c J=\ln(\sqrt{2}+1)/2=0.440\,687...$. The value of the limit of both sequences of points on the vertical axis is the exponent $\eta=\frac{1}{4}$.

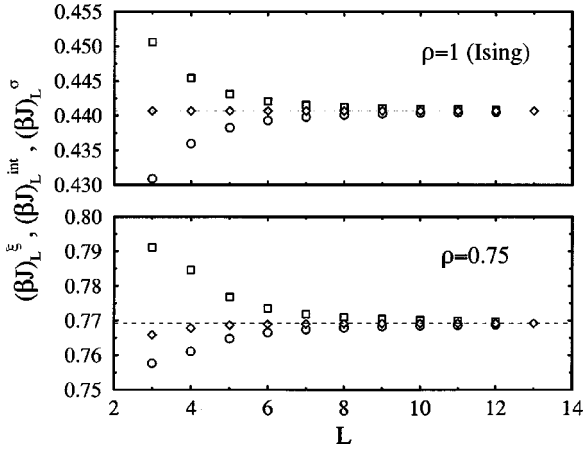

FIG. 2. Sequences of finite-size approximations to the critical inverse temperature $(\beta J)_L^{\xi}$ (circles), $(\beta J)_L^{\sigma}$ (squares), and $(\beta J)_L^{\rm int}$ (diamonds) as a function of strip width L for $\rho=1$ (the pure Ising case) and for spin density $\rho=0.75$. It is apparent that in both cases the last sequence converges faster than the other two to the critical inverse temperature of the infinite system represented by the dashed line. Here, as well as in the following pictures, no error bars are shown if they are smaller than the symbol's size.

attractive for a numerical study, i.e., the possibility of studying its critical behavior by sitting exactly at T_c. A very precise numerical determination of T_c is thus called for in our case, and it has indeed been reached by a joint analysis of the finite-size behavior of the correlation length [as defined in Eq. (8)] and of the *domain-wall free energy* (or *interfacial tension*), the latter obtained by comparing free energies of systems with periodic and antiperiodic boundary conditions in the direction perpendicular to the strip. Antiperiodic boundary conditions in a ferromagnetic system force an interface along the cylinder's length. In practice, the interface is created by introducing an antiferromagnetic seam along the cylinder, i.e., by reversing in each row the bond $J\rightarrow -J$ between two fixed spins (e.g., spins L and 1). The interface (domain-wall) free energy per unit length σ_L is given by the difference in free energy between the system with periodic and antiperiodic boundary conditions, and reads

$$\beta\sigma_L=-\ln\frac{\gamma_{1,abc}}{\gamma_1} \tag{12}$$

with $\gamma_{1,abc}$ the largest eigenvalue of the TM with antiperiodic boundary conditions [compare Eq. (12) to Eq. (8)].

Phenomenological renormalization [29,30] predicts that the correlation length ξ_L scales as L at criticality. The correct formula reads

$$\xi_L^{-1}=L^{-1}(A+B_{\xi}L^{y_{\rm irr}}+\cdots), \tag{13}$$

where the correction-to-scaling terms (the exponent $y_{\rm irr}$ is related to the presence of irrelevant scaling fields and it is known to be $y_{\rm irr}=-2$ for the Ising model [35]) split the exact crossing of the curves L/ξ_L versus βJ at the critical point for different values of L into a sequence of distinct intersections of abscissas $(\beta J)_L^{\xi}$ (see Fig. 1). Extrapolating this sequence to its asymptotic value provides an accurate determination of the critical temperature. The task is accomplished by a combination of different extrapolation techniques. They are basically: the Bulirsch and Stoer algorithm discussed by Henkel and Schütz [36], the three-point iterated fit method presented by Blöte and Nienhuis [37], and a third method, close in spirit to the latter, that uses fitting procedures at all stages of extrapolation [38]. These three different algorithms give accurate and consistent results, since they lead to extrapolated values that do not significantly differ from each other. They are also used in the following wherever the $L\rightarrow\infty$ limit of a sequence needs to be extracted. Their comparison sets the error bars of our analysis.

The same scaling argument presented for ξ_L^{-1} can be applied also to the wall free energy $\beta\sigma_L\sim L^{-1}$, and the corresponding sequence $(\beta J)_L^{\sigma}$ similarly analyzed.

The apparent ''mirror symmetry'' of the two sets of curves displayed in Fig. 1 suggests also a new method by which to extract $\beta_c J$: the analysis of the sequence $(\beta J)_L^{\rm int}$ of mutual intersection points of the curves L/ξ_L and $L\beta\sigma_L$ versus βJ for fixed values of L. This sequence also converges to $\beta_c J$ in the limit of large L, but *much* faster than the previous sequences $(\beta J)_L^{\xi}$ and $(\beta J)_L^{\sigma}$, and its extrapolation through the methods described above furnishes a much more precise determination of the critical temperature (upper half of Fig. 2). Already for the rather small value $L=6$, the pure 2D Ising model value of $(\beta J)_L^{\rm int}$ differs from the exact critical value only after the 11th decimal digit. Neither this striking result nor the analysis of the $(\beta J)_L^{\rm int}$ sequence are, to our knowledge, present in the literature, and we regard them as remarkable findings on their own.

The whole machinery can be carried over to the diluted case. It is worth remembering that in our investigation the spin density ρ is treated as a parameter fixed to definite values in the range $\rho_c\leqslant\rho\leqslant 1$ to scan the phase diagram. The values of ρ analyzed are $\rho=0.95$, $\frac{8}{9}$, 0.8, 0.75, $\frac{2}{3}$.

For $\rho\neq 1$ the mirror symmetry of Fig. 1 is lost when the value of ρ substantially departs from the pure system value. Logarithmic terms would also appear in Eqs. (13) in the case

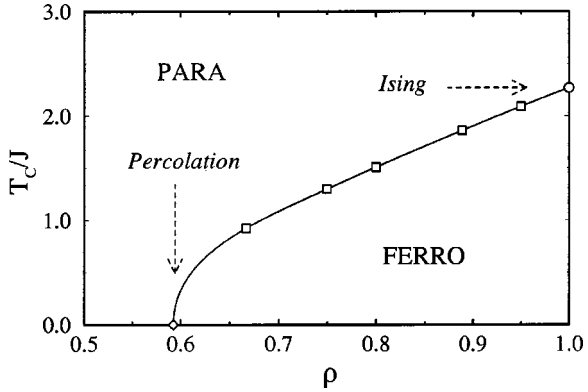

FIG. 3. Phase diagram of the 2D site-diluted Ising model, with its ferromagnetic and paramagnetic phases. The diamond and the circle denote, respectively, the exact percolation threshold and the transition temperature of the pure Ising model; the squares correspond to the values of spin density ρ analyzed in this investigation. The line is merely a guide to the eye.

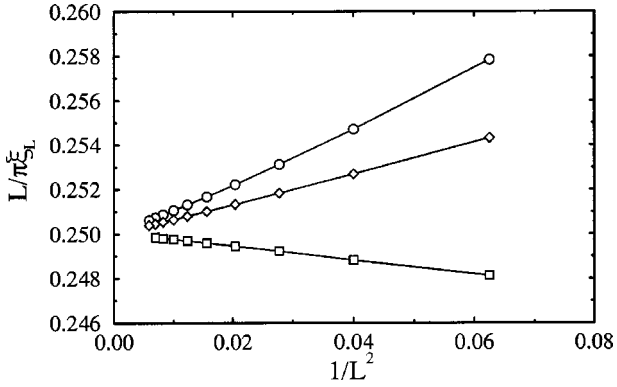

FIG. 4. Scaled inverse correlation length $L/\pi\xi_L$ plotted as a function of $1/L^2$ for the pure Ising case (circles), for $\rho=0.80$ (diamonds), and for $\rho=\frac{2}{3}$ (squares). The lines just connect the displayed points. Note the vertical scale.

of the logarithmic-corrections scenario [32] (see the Appendix and the discussion in Sec. IV B below). In spite of that, the sequences $(\beta J)_L^{\mathrm{int}}$ still converge quite rapidly to their asymptotic value, again allowing extremely accurate determination of $(\beta_c J)(\rho)$ (see lower half of Fig. 2) also for the lower values of ρ, where closeness to the percolation threshold induces stronger finite-size deviations from the asymptotic behavior of the quantities under investigation.

Figure 3 presents the phase diagram of the 2D site diluted Ising model, showing the values of $T_c(\rho)/J$ obtained in this way for strip widths up to $L=13$.

The critical slope of the curve $T_c(\rho)$ versus ρ at $\rho=1$ is exactly known [39] to be $S_c=T_c^{-1}dT_c/d\rho|_{\rho=1}=2/[\ln(1+\sqrt{2})(1+\sqrt{2}/\pi)]=1.564\,785...$; from an additional calculation limited to size $L=11$ at a value of ρ very close to 1 and from a numerical evaluation of the derivative, we obtain an estimate that differs from the exact result by approximately 0.01%.

Finally, let us point out that the method just described provides a very precise determination of the critical temperatures *for each of our approximating systems*. We make no claim, however, that these values coincide with those of the fully quenched system: for fixed ρ they in fact turn out to be slightly different from system (*a*) to system (*d*) [21] and need not be the same as Monte Carlo data [14]. The reason is that nonuniversal quantities, such as indeed the critical temperature or the value of the percolation threshold ρ_c, do depend on the approximating system chosen, while universal quantities, such as critical exponents, central charge, etc...., do not. Our confidence in this statement, and consequently in the validity of our method, will be addressed in Sec. V.

B. The exponent η and the ratio γ/ν

Having determined with great precision the critical temperature of our model, we now turn to the evaluation of critical exponents.

The theory of conformal invariance [40] relates the amplitude A of the leading term in the FSS behavior of ξ [Eq. (13)] to the anomalous dimension of the spin-spin correlation function via $A=\pi\eta$, providing a simple method by which to extract η just by extrapolating the sequence L/ξ_L to $L\to\infty$. The same procedure can be applied as well to $L\beta\sigma_L$, since the amplitudes for the correlation length and for the wall free energy are equal.

In the pure Ising case, η is known to be $\frac{1}{4}$. Our extrapolations unambiguously give the same value

$$\eta=\frac{1}{4} \tag{14}$$

for all the values of ρ considered, with a maximum estimated error of 0.0004 (the error arising from the uncertainty in the location of the critical temperature being much smaller than the difference between final values from different extrapolation procedures). From this we can conclude that the exponent η does not depend on disorder, a result that is already known [20,21,9,41,11] but without the precision shown above.

Note that the correlation length ξ extracted from the ratio of TM eigenvalues [Eq. (8)] within the EEA refers to the decay of the *average* spin-spin correlation function in the disordered system. This is to be contrasted with the correlation length that describes the *typical* decay of the correlation function in a given disorder configuration, a quantity readily obtainable from a corresponding ratio of Lyapunov exponents within a random TM approach [41,11]. Our identification may be confirmed by plotting L/ξ_L versus $1/L^2$ (Fig. 4) and noting that no corrections to a linear dependence smoothly extrapolating to $\frac{1}{4}$ appear, as it could instead be expected for the behavior of the typical correlation length (see in particular the discussion focusing on Fig. 1 of Ref. [41], and Ref. [11]).

We would like to point out that the smallness of the estimated error of the exponent η is of particular relevance. It can in fact also be read as strong confirmation that our approximations are sufficient to place the model under study in the right universality class, and that universal quantities therefore turn out to be correctly reproduced.

This is confirmed by the behavior of the magnetic susceptibility data obtained via the group-theoretical-reduction technique. The finite-size scaling analysis of the susceptibility χ predicts a leading L dependence of the form $\chi_L\sim L^{\gamma/\nu}$, and the ratio γ/ν is extracted by analyzing the sequence

$$\left(\frac{\gamma}{\nu}\right)_L = \frac{\ln(\chi_{L+1}/\chi_L)}{\ln(L+1/L)}. \tag{15}$$

Corrections to scaling generally modify the pure power law, and an extrapolation to $L\to\infty$ is needed. Corrections may again be of pure power-law form in L (in the conventional situation arising from irrelevant scaling fields), or indeed by terms containing logarithms of L (in the log-corrections scenario). In the latter case, if the largest available system size is too small, the data might still reflect a preasymptotic regime, and the very idea of extrapolating a sequence like Eq. (15) may lose its validity in that case. We refer the reader to the Appendix for a discussion of this point. We are, however, not able to detect the nonuniversal corrections predicted to be present in the preasymptotic regime of the log-corrections scenario. They are either absent, or too small to be detected (Eq. A16d). The same is true for η.

Extrapolating the sequence (15) provides the result

$$\frac{\gamma}{\nu} = \frac{7}{4} \tag{16}$$

for all the points studied in the present paper. The estimated error is around 0.001, bigger than for η because a shorter sequence ($L_{\max}=9$) has been utilized. These data show that the ratio γ/ν is independent of the degree of dilution, a finding in complete accord with the constancy of η via the Fisher relation $\gamma/\nu = 2-\eta$, which is thereby demonstrated to be satisfied with a precision of the order of a tenth of a percent.

The observed constancy of η and γ/ν is unfortunately not sufficient to discriminate between the two admissible scenarios, since *both* predict that the ratio retains the pure Ising value $\frac{7}{4}$, irrespective of disorder. It only shows that for these two quantities extrapolations are not noticeably misguided by logarithmic corrections, in case they are present.

C. The central charge

Conformal invariance [42] provides a link between the values of the dimensionless free energy per site at critically f_L in a finite strip with periodic boundary conditions, and the central charge c which characterizes the universality class of a conformally invariant model. The relation reads

$$f_L \sim f_\infty - \frac{\pi c}{6L^2} + \cdots, \tag{17}$$

with f_∞ the nonuniversal bulk (infinite system) free energy per site. Equation (17) is supposed to retain its validity also in a random system, provided c is replaced by an *effective* central charge c' (Ref. [33]). The next-to-leading term, omitted in Eq. (17), differs depending on whether or not logarithmic corrections are present. In the latter case, it is believed to depend on L as $B_f L^{2+y_{\rm irr}} = B_f L^{-4}$ if we adopt the value $y_{\rm irr} = -2$ of Ref. [37] or, equivalently, that borne out from the analysis of Sec. IV B above. In the former case, instead, Ludwig and Cardy [33] derived the results presented in Eqs. (A18) of the Appendix. To avoid dealing with the f_∞ term, we prefer to extract finite-size estimates of c' through the formula

$$c'_L = \frac{6}{\pi}(f_{L+1}-f_L)\frac{L^2(L+1)^2}{2L+1} \tag{18}$$

and analyze them both by extrapolating the above sequence to c', and possibly looking for finite-size corrections (next-to-leading terms).

The outcome of our analysis does not differ substantially from that which we obtained in the study of the exponent η. That is, while c' appears to be clearly given by

$$c' = \tfrac{1}{2}, \tag{19}$$

equal to the pure Ising value, for all the points studied (with a maximum error of 0.0003), we are not able to individuate any correction of the kind expressed by Eq. (A18). The plots of c'_L versus $1/L^2$ are lines showing no appreciable bending.

The estimates c'_L always converge to their asymptotic value from above, both in the pure as well as in the diluted case. This is at least a necessary condition for the reflection positivity property, which holds for unitary models but might not hold for random models. The discussion on this point in Ref. [43] is indeed based on Eqs. (A18), which would predict a convergence from below, the first correction terms to the value $c'=\frac{1}{2}$ being negative. But again, they are either absent or too small to be discernible within our errors, or convergence from below sets in for system sizes beyond those accessible through our TM calculations.

D. The analysis of the correlation length

The correlation length provides us with another tool with which to push our investigation further: the study of the exponent ν, extracted, in addition to η, from a FSS analysis of ξ, can be of particular relevance since it is a single isolated exponent and is therefore predicted to show a different behavior in the two scenarios.

Our analysis follows general reasoning [29]. A recent application to disordered Ising models has been provided by Aarão Reis *et al.* [12]. Two quantities at our disposal can be subjected to the same investigation, the correlation length ξ as well as the domain free energy $\beta\sigma$: both show the same asymptotic behavior, though with different corrections to scaling.

First we define the derivatives of these quantities with respect to temperature:

$$\mu(t) = \frac{d\xi(t)}{dt}, \tag{20}$$

$$\mu'(t) = \frac{d[\beta\sigma(t)]^{-1}}{dt}. \tag{21}$$

A pure power-law divergence of $\xi(t)$ at critically like $t^{-\nu}$ implies $\mu(t)\sim t^{-\nu-1}$, which in turn translates into the finite-size prediction $\mu_L \sim L^{1+1/\nu}$. The same relations hold respectively for $\beta\sigma(t)$, $\mu'(t)$ and μ'_L. From the μ_L sequence (and similarly form the μ'_L sequence), one obtains finite-size approximations ν_L of the correlation length exponent via

$$\nu_L^{-1} = \frac{\ln(\mu_{L+1}/\mu_L)}{\ln(L+1/L)} - 1. \tag{22}$$

TABLE I. Various exponents for different values of spin density ρ. In the second column is the exponent ν as extrapolated from Eq. (22). The third column shows the value of the ratio α/ν extracted from Eq. (25). The last column provides a test for the validity of the hyperscaling relation $2/\nu-\alpha/\nu=d=2$.

ρ	ν	$\frac{\alpha}{\nu}$	$\frac{2}{\nu}-\frac{\alpha}{\nu}$
0.95	1.054±0.002	−0.102±0.0088	1.999±0.012
8/9	1.113±0.001	−0.214±0.007	2.011±0.009
0.80	1.15 ±0.03	−0.30 ±0.02	2.04 ±0.06
0.75	1.18 ±0.02	−0.35 ±0.02	2.05 ±0.05
2/3	1.23 ±0.05	−0.445±0.005	2.07 ±0.07

Extrapolated values clearly appear to vary with ρ, and to change continuously from the Ising value $\nu=1$ to the percolation value $\nu=4/3$ [44] in a way shown in Table I. The large error bars on the values corresponding to stronger dilution are due to degradation in the quality of the fits, and even to the appearance of nonmonotonicities in the sequence ν_L versus L extracted from the domain-wall free energy data, which prevent an extrapolation as precise as those for larger ρ values. The cause is probably the closeness of the percolation threshold, which requires a corresponding enlargement in system size in order to disentangle the system's correct critical behavior from crossover effects.

A similar data trend was extracted also from Monte Carlo simulations, [20] and analogous variation with the strength of bond disorder was observed in Ref. [12]. The variation of ν as a function of ρ at constant η and constant γ/ν would qualify the observed nonuniversality—if it must be upheld—to be of the *weak* form.

On the other hand, one should be wary of the fact that preasymptotic logarithmic corrections might invalidate the very idea of extrapolating a ν_L sequence of limited length; see Eq. (A16c). This was the attitude advocated by Aarão Reis *et al.*, who did not give much weight to an analogous variation of *their* extrapolated exponent ν as a function of disorder. Rather, they proposed to fit the L-dependent μ_L data to a form expected to hold in an intermediate (preasymptotic) range of L values within the logarithmic corrections scenario, viz.

$$\mu_L \sim L^2(1-A\ln L)^{1/2}, \tag{23}$$

which predicts $(\mu_L/L^2)^2$ to be linear in $\ln L$. Note that this expression used by Aarão Reis *et al.* was derived from heuristic considerations. It formally agrees with the true expression (A16b) only to first order in $\ln L$, but it predicts the *wrong* sign for the coefficient of the leading $\ln L$ behavior of the effective size-dependent correlation length exponent ν_L [Eq. (A16c)].

In Fig. 5 we plot our results in the same form as Aarão Reis *et al.* do, both for μ_L and for μ_L'. The data show exactly the same trend observed in Ref. [12], both with respect to system size L and with respect to the strength of the disorder. From this observation we derive additional strong confidence in the validity of our method. The pure system behavior is soon replaced by an apparently linear term in $\ln L$ for $(\mu_L/L^2)^2$ in the increasingly disordered system, as

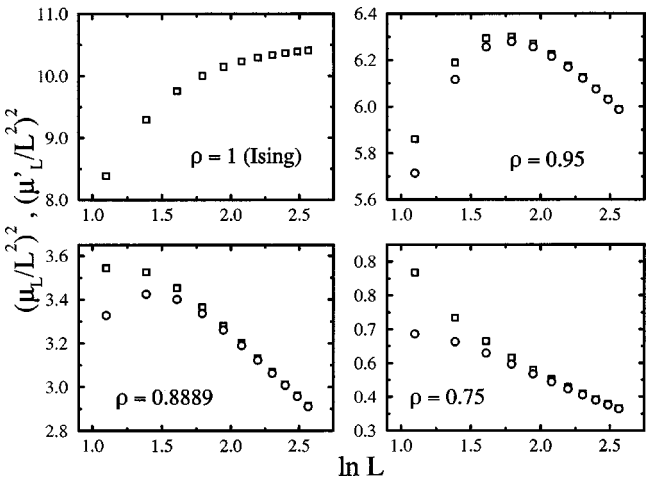

FIG. 5. Quantities $(\mu_L/L^2)^2$ $(\mu_L'/L^2)^2$ at criticality plotted as a function of $\ln L$, for various values of ρ.

would be expected from Eq. (A16b). Note once more the same asymptotic behavior of μ_L and μ_L', with different corrections to scaling.

The data can, however, be consistently interpreted within a scenario of nonuniversal critical exponents as well. Indeed, for a power-law divergence of the correlation length with $\nu>1$, FSS would predict $(\mu_L/L^2)^2 \sim L^{-\omega}$ with $\omega=2-2/\nu$ at sufficiently large L for the data plotted in Fig. 5. In the disorder range considered, ω is a rather small quantity since the ν extrapolated from Eq. (22) only slightly exceed 1. The data might therefore easily look to be exhibiting a $(1-A\ln L)$ behavior for the accessible range of system sizes, which was the form assumed in Ref. [12]. This might, however, just be the result of an expansion to first order in the small quantity $\omega\ln L$ $(L^{-\omega}\sim 1-\omega\ln L)$. Upon closer inspection, a curvature compatible with an $L^{-\omega}$ behavior is indeed discernible in the figure for larger disorder. Using a fit against $L^{-\omega}$ we have another way of analyzing our data to determine ν, and the values obtained this way are compatible with those obtained from extrapolation of the ν_L data. Nonetheless, a curvature of the $(\mu_L/L^2)^2$ data may occur also in the log-corrections scenario [depending on the value of D_1 in Eq. (A13)].

We can, however, affirm with certainty that neither Aarão Reis *et al.* nor we are seeing preasymptotic effects of logarithmic corrections in the data for ν_L, because they are *increasing* with L (in our case at least for $\rho>0.8$), whereas according to FSS they should *decrease*; see Eq. (A16c). Thus, *if* logarithmic corrections are present, they are for the available system sizes still masked by other corrections to scaling. Again, it seems that further investigation is required in order to be able to take sides on the critical behavior of this system.

E. The specific heat

The logarithmic-correction scenario predicts for the specific heat the double-logarithmic divergence of Eq. (11). This translates, at criticality, into the FSS prediction [9]

$$C_L \sim C_0' + C_1' \ln(1+C_2'\ln L), \tag{24}$$

where the pure-system critical behavior is recovered by the

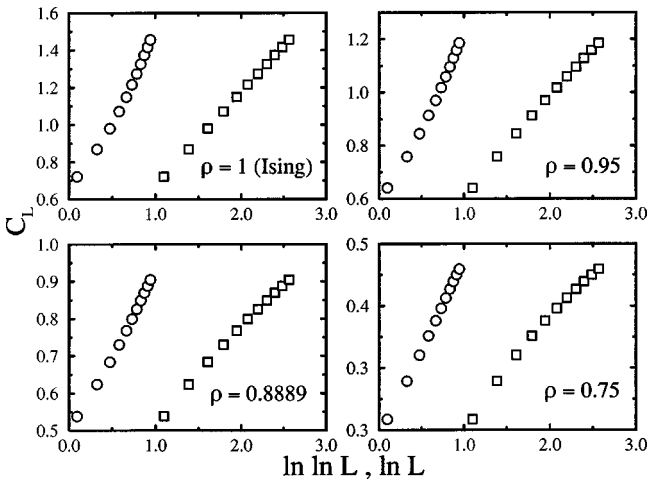

FIG. 6. Finite-size approximants of the specific heat at criticality C_L versus $\ln\ln L$ (circles) and $\ln L$ (squares).

vanishing of C_2' (but with $C_1'C_2'=\text{const}$) for $\rho=1$. On the other hand, a scenario of nonuniversal critical exponents would lead to

$$C_L \sim C_\infty + C_3' L^{\alpha/\nu}. \tag{25}$$

Though α/ν is a ratio of exponents, it is expected to vary with ρ, being directly related to ν via the hyperscaling relation $2-\alpha=d\nu$, with d the system's dimensionality. Moreover, this relation imposes a severe constraint on α: since ν is found to be greater than 1, α should be smaller than 0, in other words the specific heat should then turn out to be nondivergent, with its finite-size estimates saturating to a value C_∞ at T_c.

We plot our data for the specific heat in the same form as in Ref. [13], i.e., against both $\ln L$ and $\ln\ln L$ (Fig. 6). The pure-system divergence is well reproduced by the straight line the data show when plotted against $\ln L$ and, equivalently, by the upward curvature when plotted against $\ln\ln L$. Exactly as it happens in Ref. [13], as disorder is switched on and increased, it is apparent that both curves tend to bend downward, the former markedly deviating from, the latter instead approaching, a straight line (from top left to bottom right of Fig. 6). The law describing these finite-size data therefore has to be at least less divergent with L than $\ln L$. However, the same kind of behavior must at moderate system sizes be expected in *both* scenarios (as we shall argue in greater detail in the concluding section). It is therefore difficult to conclude definitely in favor of Eq. (24): here more than elsewhere the need for significantly larger system sizes is particularly felt.

If we fit the data to a nondiverging size dependence, in accordance with Eq. (25), there is a consistency check to carry out, i.e., one may ask oneself whether or not the hyperscaling relation is verified, given the ν values from the correlation length. The fitting of the specific heat to the form (25) is performed first by taking all the points into account, then by reducing the data set by successively increasing the initial value of L considered. In this way one can both check for the stability of such a fit and get a sequence of values of α/ν that may be subjected to subsequent extrapolation procedures when needed. The results of this analysis are shown in Table I. Its main outcome is the fact that, within a weak-universality interpretation of the data, a small, seemingly systematic offset from hyperscaling though still within estimated error bars, appears.

We have tried to discriminate between Eqs. (24) and (25) on the basis of a χ^2 test. Although such a test appears to favor the log-corrections scenario by roughly an order of magnitude in χ^2, the χ^2 values for both alternatives are so small [typically $\mathcal{O}(10^{-8})$ and $\mathcal{O}(10^{-7})$, respectively, for a choice of the data set] as to make it doubtful whether a meaningful model selection should be based on them. In particular, our data are not normally distributed random data (indeed, they are deterministic), as instead strictly required by a reliable χ^2 analysis.

V. DISCUSSION AND CONCLUSIONS

In this paper we studied the critical behavior of the 2D Ising model with quenched random site dilution via the EEA. Our purpose was twofold: first we wanted to address again the question about the reliability of the method; second, we extended the study begun in Ref. [21] to systems of larger sizes, also in view of the new results which have appeared in the literature since then [12–16], in order to enable perhaps a clearer discrimination between the two contradictory pictures of the system's critical behavior that have survived so far, the logarithmic corrections and the weak-universality scenario.

A. Reliability of the method

Since the description of quenched disorder within our EEA is only approximate, we have to worry about how good it actually is. Before gathering the different pieces of evidence accumulated during our numerical study confirming the validity of our approach, we would like to address the question from a more general viewpoint. In Ref. [21] the correlation length ξ_k describing the asymptotic decay of the correlation function $G_k(r_{ij})$ of the *disorder* degrees of freedom was introduced and studied. Its finite-size estimates can be computed from the ratio of two TM eigenvalues in complete analogy to Eq. (8) by replacing γ_2 with $\tilde{\gamma}_2$, where $\tilde{\gamma}_2$ is the second eigenvalue of the symmetric block of the transfer matrix Γ.

As regards the analysis of this new correlation length, we have both bad and good news. The bad news is that ξ_k actually diverges at criticality in our different approximating systems. This escaped our attention in Ref. [21]: while ξ_k was observed never to exceed a few lattice spacings for the accessible system size, its size dependence at T_c had not been monitored. However, before concluding from this that our systems provide only a rather poor description of quenched disorder, we have also determined the behavior of the ratio $R_L=\xi_{k,L}(T_c)/\xi_L(T_c)$ of these two correlation lengths at criticality, and the results of this study may be taken for the good news. We find that $R_L \to \frac{1}{8}$ at large L, independently of ρ. Since this limit represents nothing but the ratio η/η_k of the anomalous dimensions of the spin and occupation-variable correlation functions at T_c, this implies that $\eta_k=2$, hence is unusually large. The k_i correlator is thus ''almost summable'' at criticality (its sum has a logarithmic

infrared divergence). Note that the ratio $G_k(r)/G(r)$ behaves as

$$\frac{G_k(r)}{G(r)} \sim \frac{1}{r^{7/4}} \tag{26}$$

and thus decays to zero, stating that k_i correlations are negligibly small if compared to σ_i correlations at large distances. This is why the system may be regarded as effectively quenched despite remaining correlations between the k_i, and it may be regarded as the reason why our results compare so favorably with those obtained via more conventional approaches.

Indeed, both checks against exact results, wherever obtainable, and direct comparison of our data with those obtained by the random TM approach (which allows for an in-principle exact treatment of the disorder) increase our confidence in the validity of our method beyond any reasonable doubt. Among the exact results correctly reproduced by our method we mention (i) data obtained for a one-dimensional system [45], (ii) the value of the initial slope of the critical line in the phase diagram, reproduced to within 0.01%; (iii) the correct value of the connectivity length exponent ν_p and of the crossover exponent $\phi = \nu_p/\nu = 1$ at percolation [21] (iv) the precision with which the values of the exponent η, of the ratio γ/ν, and of the central charge are determined for all the points investigated in the phase diagram. In addition, as mentioned above, the random TM data (obtained in the case of bond-disordered systems) [12,13] exhibit the same finite-size signature, and qualitatively the same behavior as regards their dependence on the disorder strength as those obtained in the present paper.

A definite advantage of our method over other numerical methods dealing with disordered systems is that it does not suffer from non-self-averaging difficulties, and provides directly through the ratio of TM eigenvalues the average correlation length, not the typical one, such that the successive analysis turns out to be more straightforward than, e.g., in Ref. [12].

The results discussed so far were all obtained within approximating system (d). Comparative studies including results also from the other systems $(a)-(c)$ were presented in Ref. [21]. They showed that systems $(a)-(d)$ appear to be in the same universality class as regards their critical behavior. Nonuniversal quantities, however, such as the critical temperature itself, or the value of the percolation threshold ρ_c, do depend on the approximating system, differing between those with and without the plaquette constraint and those estimated by other numerical works [14,34]. The conclusion is that the description of quenched disorder, though only approximate, appears to be precise enough to put all our approximating systems into the same universality class as the fully quenched system, and this is all that is required for the purpose of the present study.

B. Discriminating between the scenarios

We now come to our second point, concerning the discrimination between the two scenarios proposed for the critical behavior of the model considered in this paper. Our view is that some of the FSS investigations that have appeared in the literature—including the recent ones [8,10,12–14,16,20,21]—may perhaps not be as decisive on this matter as their authors have tended to believe. We shall now discuss some issues on which, we believe, clarification or amendments are needed, presenting a critical review of available material in the light of our own data analysis.

The results of Refs. [20] and [21] have been interpreted as providing evidence in favor of a weak-universality scenario. The former is a Monte Carlo investigation both at and off criticality of the site-diluted system. On the basis of χ^2 data analyses, Kim and Patrascioiu concluded that their off-critical simulations of susceptibility and correlation length were better described by modified power laws than in terms of the log-corrections scenario. No analogous discrimination was attempted for the specific heat, and hence no check of hyperscaling was performed. The main weakness of their off-critical simulations is that reduced temperatures are still sizeable, and the constancy of the ratio γ/ν appears slightly less well satisfied than the individual errors on the values of γ and ν would allow. Simulations at criticality yielded a specific heat very slowly increasing with system size, and arguments in favor of a saturation were advanced. The constancy of γ/ν was clearly shown and the validity of the Fisher relation confirmed, but—as noted above—this cannot be taken to support either scenario over the other.

Our own earlier TM strip-scaling results [21], too, were interpreted in terms of weak universality. Possible effects of logarithmic corrections in the effective size-dependent exponents ν_L were looked for, but were either absent or simply not discernible due to the rather moderate strip widths available in that study. Indeed, the rather limited system sizes may be taken as one of the weakest points of that investigation. Moreover, specific heats had not been computed and so hyperscaling within a varying exponents picture was not checked. Again, the observed constancy of γ/ν and η may be regarded as a piece of evidence for the reliability of the method, but not in favor of either scenario, except insofar as they imply that if critical behavior in the model were nonuniversal, the observed nonuniversality would have to be of the weak form.

Let us now turn to the recent investigations that have been taken to support the log corrections scenario. They are either of the TM strip-scaling [10,12,13] or of the Monte Carlo [8,14,16] type.

The strip-scaling data of Aarão Reis *et al.* [10,12] for the correlation length in the bond-disordered system appeared to give values of ν slightly greater than that of the pure system, which the authors, however, discarded, attributing them to preasymptotic effects originating from logarithmic corrections. Their interpretation of the data in these terms does, indeed, describe the data rather well on the level of the μ_L, but *not* on the level of the size-dependent exponents ν_L. Moreover, we recall our discussion of this point in Sec. IV D, according to which the power-law picture provides a consistent interpretation of the data as well.

Turning to the critical specific-heat data presented in Refs. [12] and [13], they were interpreted as ''clearly suggesting a divergence in the thermodynamic limit.'' This point was claimed to be strengthened by pushing system size up to $L=18$ (and in less precise simulations even to an impressive $L=23$), and by noting that specific-heat data ap-

peared to be halfway between single- ($\ln L$) and double- ($\ln\ln L$) logarithmic dependence on strip width, from which a divergence in the thermodynamic limit was inferred. One should, however, note that at moderate L this kind of behavior is to be expected in *both* scenarios if the correlation-length exponent is close to 1 and, hence, α is only slightly negative. The difference is that C_L will continue to increase at least as fast as $\ln\ln L$ for *all* L in the log-corrections scenario, whereas there will be a crossover to a growth which is slower than $\ln\ln L$ at $L_* \simeq \exp\{-\nu/\alpha\}$ in the case of a modified power law (25) with $\alpha<0$. Tentatively accepting the value $\nu \simeq 1.083$ determined by Aarão Reis *et al.* [10] for the $r \equiv J_1/J_2 = 0.25$ case investigated in Ref. [13], one would have to locate the crossover length at $L_* \simeq 680$. Thus $L_{\max} = 23$ is still much too small to allow conclusion in favor of either scenario.

We have carried out χ^2 data analyses on the raw data of Ref. [13], comparing the two scenarios, as we did on our own data. Power-law fits and fits according to Eq. (24) show no significantly different quality. In particular, it seems that a double-logarithmic law provides a better fit to the data selected from a window $[L_{\min}, 23]$ with $L_{\min}$ up to 8, while power laws give a smaller χ^2 for the larger $L_{\min}$ up to 13; for still larger L the results of the fits become questionable for both hypotheses. Other kinds of fits have been tried, e.g., by selecting a movable window $[L_{\min}, L_{\min}+7]$ and letting $L_{\min}$ run over the data, but without any significant improvement. Only by discarding the larger L values, which are still much too noisy (as is obvious from a quick look at the derivative information plotted in Fig. 3 of Ref. [13]), one gets values of χ^2 that slightly favor the double-logarithmic form (24). Incidentally, however, the power-law fits lead to an estimate for α/ν compatible via hyperscaling with the ν value reported in Ref. [10]. This analysis cannot thus be seen as conclusive.

The Monte Carlo study of site-diluted systems by Ballesteros *et al.* [14] shows critical specific-heat data reasonably well fitted (in terms of χ^2) by a double-logarithmic form, at least for intermediate dilution. The same is true for a recent study by Selke *et al.* [16] However, no power-law fitting is attempted for comparison. In earlier work on bond-disordered systems [7], such power-law fits had been attempted, and were regarded as inferior to double-logarithmic ones. Note, however, that the maximum system sizes studied in Ref. [7] are for $r=0.25$ still below, and for $r=0.1$ above but still very close, on a double-logarithmic scale, to the crossover lengths L_* expected from the ν values reported in Ref. [10].

The ν_L data presented in Ref. [14] have, in our view, error bars somewhat too large to allow concluding with confidence that $\nu_L \to 1$ for large L for all the values of ρ also because Ballesteros *et al. force* the intercept through 1 rather than fitting it. The reasonably large values of L reached in this investigation would probably put the system in the asymptotic regime (at least for the stronger disorder), and the FSS expression used by Ballesteros *et al.* is indeed $\nu_L = 1 + A'/\ln L$. Nonetheless, the constant A' in this expression should, strictly speaking, come out to be $\frac{1}{2}$, *independently* of the degree of disorder [see Eq. (A17c)], a property which their data and their fits do *not* respect.

In the off-critical Monte Carlo study of Talapov and Shchur [8], bond disorder was observed to lead to increased values for the magnetization and susceptibility exponents β and γ, while the behavior of the specific heat showed good agreement with the double-logarithmic form (11). Together with the Rushbrooke equality $\alpha + 2\beta + \gamma = 2$, these findings embody an inconsistency from which the authors concluded that their increased values for β and γ cannot be asymptotic. On the other hand, it turns out that the specific-heat data of Ref. [8] can be fitted equally well to a power-law form with a value of α compatible via the Rushbrooke relation with the increased values of β and γ [46]. Note also that the ratio β/γ in the disordered system is the same (to within a fraction of a percent) as in the pure system, as would be expected in a weak universality scenario, whereas individually β and γ change in the 6% range.

The difficulties of FSS are avoided in the series expansion study of Roder *et al.* [15], their determination of the exponent $\tilde{\gamma}$ in Eq. (10), which appears to saturate at $\frac{7}{8}$ for sufficiently strong disorder, may perhaps be taken as the strongest piece of evidence currently available in favor of the validity of the logarithmic-corrections scenario. Still, the analysis suffers from the relatively small length of the series, at least in some regions of the phase diagram, and one would wish this to be made more conclusive by including higher-order terms, so as to reduce error bars. It is also worth noting at this point that their claim that $\chi_L(T_c)$ is unaffected by logarithmic corrections cannot be upheld: see Eq. (A15).

We now turn to the present investigation. First, by combining finite-size signatures of correlation length and domain-wall free energy, we have in particular been able to locate critical temperatures with extreme precision. Second, we have significantly enlarged our system sizes. In interpreting our data, maximum care was constantly taken to be open to both possibilities. Based on results of Cardy and Ludwig, a more systematic FSS analysis than in previous numerical studies has been performed. As we will presently show, neither way of looking at the available finite-size data is completely satisfactory.

The value of the effective central charge is found to be $c' = \frac{1}{2}$ with extremely high precision. If conformal invariance and reflection positivity also held for disordered systems, this finding would put the model into the Ising universality class and exclude continuously varying exponents [47]. However, the results of Ludwig and Cardy [33] imply that the latter condition does *not* hold, at least for the weakly disordered ferromagnetic Ising model.

Considering the correlation-length exponent ν, we observe that our FSS estimates converge to values continuously varying with ρ. One might suspect, of course, that our extrapolations are misled because the algorithm would not pick up slowly varying logarithmic corrections. However, if we adopt this hypothesis, we find it not easy to reconcile with the fact that not even the slightest such effect is detectable in our extrapolations of η, γ/ν, and the central charge c, even though similar, albeit smaller offsets must be expected to occur in these quantities as well, as is borne out by our analysis of FSS in the Appendix.

The fact that the critical specific-heat data may be well fitted both to a double-logarithmic behavior (24) and to a cusp singularity (25) shows that it is difficult to discriminate with confidence between the two laws. However, we recall that a fit according to a modified power law (25) entails a

weak violation of hyperscaling, though it is still within error bars.

In conclusion, as regards the two issues raised in this paper, we believe to have provided if not a proof, at least satisfying arguments in favor of the trustworthiness and accuracy of the EEA to disordered systems. As to the second issue, we feel that further investigation would still be needed to provide clearer and definite evidence in favor of either picture. The problems seen in the FSS analyses of the system are not exclusive to our results, but are—as we believe to have demonstrated—common to virtually all previous FSS data obtained so far.

Let us finally emphasize that we have no reasons to *a priori* distrust the correctness of the theoretical picture that began to emerge through the work of Dotsenko and Dotsenko [2] and the improved and corrected versions of Shalaev [3], Shankar [4], and Ludwig [5]. Still, in an ideal world, one would like to see this picture supported by numerical evidence much better than that which is currently available.

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APPENDIX: FSS WITH LOGARITHMIC CORRECTIONS

In this Appendix, we collect the main results of the renormalization-group analysis for the case where small amounts of disorder constitute a marginally irrelevant perturbation at the pure system's fixed point, thus giving rise to logarithmic corrections, as in the field-theoretical approach to the *weakly disordered* 2D Ising model. We do this both for the sake of completeness, and to obtain a sound understanding of the effects such corrections would have in a FSS analysis. This will turn out to be relevant not only in the asymptotic regime $L \gg 1$, but especially in the *preasymptotic* regime as specified below.

We feel it particularly important to collect these results here, because the literature in the field abounds in heuristic derivations which have sometimes produced erroneous results and still more frequently even misconceptions as to what the effect of logarithmic corrections might be in the FSS signature of various thermodynamic functions and critical exponents.

The following basically involves exploring results Ludwig and Cardy [33] obtained in a replica approach to the problem, which are in turn based on earlier results on logarithmic corrections obtained by Cardy in a more general context [32].

If g denotes the marginally irrelevant coupling arising from disorder averaging in the (replicated) Hamiltonian, and by $\{u_n\}$ the set of other scaling fields at the pure systems critical point, the RG flow equations read

$$\frac{dg}{dl} = -\pi b g^2 + \mathcal{O}(g^3), \tag{A1}$$

$$\frac{du_n}{dl} = y_n u_n - 2\pi b_n g u_n + \mathcal{O}(g^2 u_n), \tag{A2}$$

where the y_n are the eigenvalues of the linearized RG equations and the b_n are operator product expansion (OPE) coefficients, which can be formulated in terms of corresponding three-point functions [33].

Up to the order shown, these equations integrate to

$$g(l) = \frac{g_0}{1+\pi b g_0 l}, \tag{A3}$$

with $g_0 = g(0)$ and

$$u_n(l) = u_n(0) e^{y_n l} (1+\pi b g_0 l)^{-2b_n/b} \equiv u_n(0)\tilde{u}_n(l). \tag{A4}$$

Among the $\{u_n\}$, the two relevant fields are u_ε which couples to the energy density and has RG eigenvalue $y_\varepsilon = 1$, and u_σ coupling to the magnetization density with $y_\sigma = \frac{15}{8}$. One may thus put $u_\varepsilon(0) = t$ and $u_\sigma(0) = h$. In a finite system of linear extent L one may formally interpret L^{-1} as another relevant scaling field with RG eigenvalue 1.

In the context of the replica formulation, the OPE coefficients and the coupling g are normalized such that the zero-replica limit of the product bg_0 and of the ratios b_n/b exist and are finite. The results of Ludwig and Cardy imply $bg_0 = 8\Delta$ and $b_\varepsilon/b = \frac{1}{4}$ in this limit, whereas $b_\sigma/b = 0$. Here Δ denotes the (bare) disorder strength: $\Delta \propto \rho(1-\rho)$ for a randomly site-diluted model

The RG equations for the singular part of the free energy and the correlation length read

$$f(t,h,g_0,L^{-1}) = e^{-dl} f(u_\varepsilon(l), u_\sigma(l), g(l), e^l L^{-1}) \tag{A5}$$

and

$$\xi(t,h,g_0,L^{-1}) = e^l \xi(u_\varepsilon(l), u_\sigma(l), g(l), e^l L^{-1}). \tag{A6}$$

Irrelevant scaling fields providing additional corrections to scaling have for simplicity been suppressed in these expressions. As usual, critical behavior characteristics of the infinite system are obtained by considering f and ξ and their derivatives with respect to temperature (and field) at a scale l chosen such that $u_\varepsilon(l) = \pm 1$. For $|t| \ll 1$ this gives

$$e^{y_\varepsilon l} = \frac{1}{|t|}\left(1 + \frac{\pi b g_0}{y_\varepsilon} \ln\frac{1}{|t|}\right)^{2b_\varepsilon/b}. \tag{A7}$$

Inserting the values for the OPE coefficients reported in Ref. [33], and $\nu = 1/y_\varepsilon = 1$ for the pure 2D Ising model, this gives the divergence of the correlation length at $h = 0$ to leading order,

$$\xi(t) \sim |t|^{-1}\left(1 + 8\pi\Delta \ln\frac{1}{|t|}\right)^{1/2}. \tag{A8}$$

The fact that $b_\sigma/b=0$ entails that magnetization and susceptibility will *to leading order* scale as pure powers of the correlation length, in agreement with the earlier results of Shalaev [3].

Phenomenological renormalization is based on analyzing the finite-size signatures of Eq. (A6) and of its temperature derivative

$$\mu_L(t)\equiv\frac{d}{dt}\xi(t,h,g_0,L^{-1}) =e^l\tilde{u}_\varepsilon(l)\xi_\varepsilon(u_\varepsilon(l),u_\sigma(l),g(l),e^lL^{-1}), \quad \text{(A9)}$$

where ξ_ε designates the partial derivative of the correlation-length scaling function (A6) with respect to $u_\varepsilon(l)$. Finite-size signatures at criticality ($t=0$, $h=0$) are obtained by analyzing these quantities at a scale $e^l=L$, giving

$$\xi_L^{-1}=L^{-1}\xi^{-1}(0,0,g(\ln L),1)=L^{-1}\Phi(g(\ln L)) \quad \text{(A10)}$$

and

$$\mu_L=L^2(1+8\pi\Delta\ln L)^{-1/2}\xi_\varepsilon(0,0,g(\ln L),1) =L^2(1+8\pi\Delta\ln L)^{-1/2}\Phi_\varepsilon(g(\ln L)), \quad \text{(A11)}$$

respectively. Equations (A10) and (A11) define the universal scaling functions $\Phi(x)$ and $\Phi_\varepsilon(x)$ of the argument x which, in the zero-replica limit, is $x=\Delta/(1+8\pi\Delta\ln L)$. In the weak-disorder limit or at large L this quantity is small, and one assumes that an expansion of these scaling functions in x exists. For Φ one gets $\Phi(x)=\phi_0+\phi_1x+\phi_2x^2+\mathcal{O}(x^3)$, the coefficients being known to be $\phi_0=\pi\eta$ from conformal invariance [40], and $\phi_1=0$ on account of the vanishing of b_σ. Thus

$$\xi_L^{-1}=L^{-1}\left\{\pi\eta+\phi_2\left[\left(\frac{\Delta}{1+8\pi\Delta\ln L}\right)^2\right] +\mathcal{O}\left[\left(\frac{\Delta}{1+8\pi\Delta\ln L}\right)^3\right]\right\}. \quad \text{(A12)}$$

A similar expansion is expected to hold for Φ_ε, i.e., $\Phi_\varepsilon(x)=\phi_{\varepsilon0}+\phi_{\varepsilon1}x+\mathcal{O}(x^2)$, so (with $D_1=\phi_{\varepsilon1}/\phi_{\varepsilon0}$)

$$\mu_L=\phi_{\varepsilon0}L^2(1+8\pi\Delta\ln L)^{-1/2}\left\{1+\frac{D_1\Delta}{1+8\pi\Delta\ln L} +\mathcal{O}\left[\left(\frac{\Delta}{1+8\pi\Delta\ln L}\right)^2\right]\right\}, \quad \text{(A13)}$$

though the coefficients $\phi_{\varepsilon0}$ and $\phi_{\varepsilon1}$ are to the best of our knowledge not known. Interest in this quantity stems from the fact that, within the phenomenological renormalization-group scheme, the correlation-length exponent ν is obtained by computing the sequence of finite-size approximants ν_L, defined by

$$\nu_L^{-1}=\frac{d\ln\mu_L}{d\ln L}-1, \quad \text{(A14)}$$

and by extrapolating this sequence to the large system limit. Equation (22) is just the finite-difference approximation to Eq. (A14).

Similarly, the FSS analysis for the susceptibility gives

$$\chi_L=L^{\gamma/\nu}f_{\sigma\sigma}(0,0,g(\ln L),1)=L^{\gamma/\nu}\Phi_\chi(g(\ln L)) \quad \text{(A15)}$$

with the same conventions for the notation. The ratio γ/ν is that of the pure system, i.e., $\gamma/\nu=\frac{7}{4}$. On account of the vanishing of b_σ, no additional logarithmic terms above those following from the expansion of Φ_x appear in Eq. (A15), in contrast to what happens for μ_L. As before, one assumes that an expansion of the form $\Phi_\chi(x)=\phi_{\chi^0}+\phi_{\chi^1}x+\phi_{\chi^2}x^2+\mathcal{O}(x^3)$ exists, and defines the effective size-dependent ratio $(\gamma/\nu)_L$ by $(\gamma/\nu)_L=\partial\ln\chi_L/\partial\ln L$. It will be useful to introduce the abbreviation $E_i=\phi_{\chi^i}/\phi_{\chi^0}$ below.

For the interpretation of numerical data it is relevant to note results for these quantities both in the preasymptotic regime $8\pi\Delta\ln L\ll1$ and in the asymptotic regime $8\pi\Delta\ln L\gg1$.

(i) In the preasymptotic regime, they read

$$\xi_L^{-1}=L^{-1}[\pi\eta+\phi_2\Delta^2(1-16\pi\Delta\ln L)] \quad \text{(A16a)}$$

$$\mu_L=\phi_{\varepsilon0}L^2[1+D_1\Delta-(1+2D_1\Delta)4\pi\Delta\ln L] \quad \text{(A16b)}$$

$$\nu_L=1+4\pi\Delta+(4\pi\Delta)^2(1+D_1/2\pi-2\ln L) \quad \text{(A16c)}$$

$$\left(\frac{\gamma}{\nu}\right)_L=\tfrac{7}{4}-8\pi\Delta^2E_1+8\pi\Delta^3(E_1^2-2E_2+16\pi E_1\ln L), \quad \text{(A16d)}$$

where we have kept the lowest order in $\Delta\ln L$. Additional terms down by further factors of Δ or $\Delta\ln L$ are not shown. These results exhibit nonuniversal corrections.

(ii) In the asymptotic regime, on the other hand, one has

$$\xi_L^{-1}=L^{-1}\left\{\pi\eta+\phi_2\left(\frac{1}{8\pi\ln L}\right)^2+\mathcal{O}\left[\left(\frac{1}{\ln L}\right)^3\right]\right\}, \quad \text{(A17a)}$$

$$\mu_L=\phi_{\varepsilon0}L^2(8\pi\Delta\ln L)^{-1/2}\left\{1+\frac{D_1}{8\pi\ln L}+\mathcal{O}\left[\left(\frac{1}{\ln L}\right)^2\right]\right\}, \quad \text{(A17b)}$$

$$\nu_L=1+\frac{1}{2\ln L}+\mathcal{O}\left[\left(\frac{1}{\ln L}\right)^2\right], \quad \text{(A17c)}$$

$$\left(\frac{\gamma}{\nu}\right)_L=\frac{7}{4}-E_1\frac{1}{8\pi(\ln L)^2}+\mathcal{O}\left[\left(\frac{1}{\ln L}\right)^3\right], \quad \text{(A17d)}$$

so the asymptotic corrections to ξ_L^{-1}, ν_L, and $(\gamma/\nu)_L$ turn out to be universal.

Ludwig and Cardy have also reported corresponding FSS expressions for the central charge, in both regimes [33]. These are

$$c=\tfrac{1}{2}-128\pi^3\Delta^3(1-24\pi\Delta\ln L) \quad \text{(A18a)}$$

and

$$c = \tfrac{1}{2} - \tfrac{1}{4}(\ln L)^{-3} + \mathcal{O}\left(\frac{1}{8\pi \ln L}\right)^4. \qquad \text{(A18b)}$$

respectively, where we have once more kept the lowest order in $\Delta \ln L$ in the preasymptotic regime.

Additional terms further down by powers of $L^{y_\alpha}(1 + 8\pi\Delta \ln L)^{-2b_\alpha/b}$, with $y_\alpha \leq -2$, will appear due to irrelevant scaling fields.

Equations (A16)–(A18) are of great importance when analyzing FSS data, if logarithmic corrections are expected to be present. Especially in a strip-scaling approach such as ours, the preasymptotic results may turn out to be of particular relevance, since the maximum size available is such that one may not reach the asymptotic regime, which is likely to be true at least for weak disorder. This has indeed been the point of view emphasized by Aarão Reis *et al.* [12].

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